

Navier–Stokes probability density function

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ABSTRACT. – A theory based on the probability density function of velocity difference, derived from the Navier-Stokes equations, is presented. The formally exact derivation leads to a pressure term and a dissipative term that need to be modeled. The asymptotic expression of the pressure term for large velocity differences can be derived using functional techniques, and a model valid for all the range of velocity differences can then be constructed. The asymptotic, large velocity difference range, of the dissipative term is modeled based on simple arguments applied to dissipative structures of Lundgren's type. Solutions of the resulting equation are studied in different types of asymptotic limits, and comparison with experimental probability distributions is made. © Elsevier, Paris

1. Introduction

One basic point in turbulence is to understand the statistical aspects of Navier-Stokes flow $\mathbf{u}(\mathbf{x}, t)$, which satisfy

$$(1) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0,$$

where p is the pressure, ρ the density assumed uniform, and ν the kinematic viscosity. In principle, the most complete statistical information is given by the probability distribution functional $\mathcal{F}[\mathbf{u}]$, determined by Hopf's equation (Hopf, 1952; Monin and Yaglom, 1971). This approach, however, poses extremely difficult mathematical problems which have so far remained unsolved. A less ambitious, and practically of much interest, aspect is described by probability distribution functions (pdf's) of different magnitudes, among which, of particularly importance is the longitudinal velocity difference across a fixed separation \mathbf{r} : $\Delta u_r \equiv (\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})) \cdot \mathbf{r}/r$. Experimentally, the corresponding pdf shows strongly non-Gaussian features (Praskovsky and Oncley, 1994; Tabeling *et al.*, 1996), some of which, such as exponential tails, have only recently been explained theoretically (Kraichnan, 1990; Castaing *et al.*, 1993; Giles, 1995). In this work we will derive an equation for the pdf of Δu_r directly from the Navier-Stokes equations (1). The formally exact derivation introduces two conditional averages which need to be modeled, one related to the pressure gradient and the other to the energy dissipation. The asymptotic expression of the pressure gradient term, valid for large values of $|\Delta u_r|$, can be calculated in closed form, which allows to derive a model valid for all Δu_r using tensor identities. The dissipation term, on the other hand, is modeled assuming that the dissipation takes place in structures of the Lundgren's type (Lundgren, 1982), and using an elementary model an expression is derived which is valid for large values of $|\Delta u_r|$. This asymptotic form allows to study the tails of the pdf which happen to have an interesting behavior as the separation r changes. In the limit $r \rightarrow \infty$, Gaussian tails exist only for $|\Delta u_r| \ll \langle (\Delta u_r)^2 \rangle^{1/2} \ln(r/\eta)$, where η is the Kolmogorov dissipation length. For larger values of $|\Delta u_r|$, always in the limit of large r , tails are exponential; finally, as r decreases, the tails become algebraic, decaying as $|\Delta u_r|^{-\lambda}$, with $\lambda \simeq 11$.

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2. Navier-Stokes pdf

The pdf of Δu_r is more conveniently derived from the joint pdf of velocity at two points. The latter can be formally written as

$$(2) \quad \Pi(\mathbf{v}, \mathbf{x}|\mathbf{v}', \mathbf{x}'|t) = \langle \Delta(\mathbf{v}, \mathbf{x}|\mathbf{v}', \mathbf{x}'|t) \rangle,$$

where $\langle \dots \rangle$ denotes ensemble average over realizations of the flow, and

$$(3) \quad \Delta(\mathbf{v}, \mathbf{x}|\mathbf{v}', \mathbf{x}'|t) \equiv \delta(\mathbf{v} - \mathbf{u}(\mathbf{x}, t)) \delta(\mathbf{v}' - \mathbf{u}(\mathbf{x}', t)).$$

Since (1), complemented with appropriate boundary conditions, is deterministic, the average over realizations can be thought of as an average over initial conditions allowing the time derivative of (2) to be written as (Lundgren, 1967)

$$(4) \quad \begin{aligned} \frac{\partial \Pi}{\partial t} = & -\nabla_{\mathbf{v}} \cdot \left\langle \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} \Delta(\mathbf{v}, \mathbf{x}|\mathbf{v}', \mathbf{x}'|t) \right\rangle \\ & - \nabla_{\mathbf{v}'} \cdot \left\langle \frac{\partial \mathbf{u}(\mathbf{x}', t)}{\partial t} \Delta(\mathbf{v}, \mathbf{x}|\mathbf{v}', \mathbf{x}'|t) \right\rangle, \end{aligned}$$

where the subindex in the ∇ operator indicates the variables on which it operates when they are other than \mathbf{x} . Use of (1) into (4) allows to write the time evolution equation for Π . The corresponding equation for the pdf of velocity differences is then obtained from (4) by changing to variables: $\mathbf{c} = \mathbf{v}' - \mathbf{v}$, $\mathbf{V} = (\mathbf{v}' + \mathbf{v})/2$, $\mathbf{r} = \mathbf{x}' - \mathbf{x}$, $\mathbf{q} = (\mathbf{x}' + \mathbf{x})/2$, and integrating over \mathbf{V} . After a direct calculation, assuming homogeneity ($\partial \langle \dots \rangle / \partial q_i = 0$), the following equation for $\mathcal{P}(\mathbf{c}, \mathbf{r}, t) = \int \Pi d^3 V$ results

$$(5) \quad \frac{\partial \mathcal{P}}{\partial t} = -\mathbf{c} \cdot \nabla \mathcal{P} - \nabla_{\mathbf{c}} \cdot (\mathbf{p} \mathcal{P}) - \frac{\partial^2}{\partial c_i \partial c_j} (D_{ij} \mathcal{P}) + 2\nu \nabla_{\mathbf{r}}^2 \mathcal{P},$$

where,

$$(6) \quad \mathbf{p} = \frac{1}{\rho} \langle \nabla p | \mathbf{c}, \mathbf{r} \rangle - \frac{1}{\rho} \langle \nabla p | -\mathbf{c}, -\mathbf{r} \rangle,$$

$$(7) \quad D_{ij} = \langle \varepsilon_{ij} | \mathbf{c}, \mathbf{r} \rangle + \langle \varepsilon_{ij} | -\mathbf{c}, -\mathbf{r} \rangle,$$

with $\varepsilon_{ij} = \nu (\partial u_s(\mathbf{x}, t) / \partial x_i) (\partial u_s(\mathbf{x}, t) / \partial x_j)$ the instantaneous dissipation tensor, and where $\langle \dots | \mathbf{c}, \mathbf{r} \rangle$ means average conditioned on given values of \mathbf{c} and \mathbf{r} , in the sense that $\mathbf{u}(\mathbf{x} + \mathbf{r}, t)$ must be equal to $\mathbf{u}(\mathbf{x}, t) + \mathbf{c}$. When no arguments are written explicitly in the magnitudes to be averaged, they are assumed to be evaluated at \mathbf{x} . Time arguments will be omitted from now on. In expressions (6) and (7) homogeneity was also used in replacing averages of the type $\langle \Phi(\mathbf{x}') | \mathbf{c}, \mathbf{r} \rangle$ by $\langle \Phi(\mathbf{x}) | -\mathbf{c}, -\mathbf{r} \rangle$, with Φ any field variable. Finally, since under reflection of coordinates axis, ∇p , \mathbf{c} , and \mathbf{r} change sign while ε_{ij} does not, one can write

$$(8) \quad \mathbf{p} = \frac{2}{\rho} \langle \nabla p | \mathbf{c}, \mathbf{r} \rangle,$$

$$(9) \quad D_{ij} = 2 \langle \varepsilon_{ij} | \mathbf{c}, \mathbf{r} \rangle.$$

(8) and (9) need to be modeled in order to close (5), which is done in the next sections.

3. Pressure term

Using Poisson equation for the pressure, expression (8) can be written in terms of the velocity field as

$$(10) \quad p_k = \int G_{ijk}(\mathbf{R}) \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) d^3 R,$$

where

$$(11) \quad G_{ijk}(\mathbf{R}) = \frac{3}{2\pi R^4} [5n_i n_j n_k - (n_i \delta_{jk} + n_j \delta_{ik} + n_k \delta_{ij})],$$

with $n_i = R_i/R$, and

$$(12) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) = \langle u_i(\mathbf{R}) u_j(\mathbf{R}) | \mathbf{c}, \mathbf{r} \rangle.$$

In (12), $\Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r})$ is the average of components i and j of the velocity field at point \mathbf{R} , conditioned on $\mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{0}) + \mathbf{c}$. To determine \mathbf{p} note that one can formally write

$$(13) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) = \frac{\int \mathcal{F}[\mathbf{u}] u_i(\mathbf{R}) u_j(\mathbf{R}) \delta(\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0}) - \mathbf{c}) \mathcal{D}\mathbf{u}}{\int \mathcal{F}[\mathbf{u}] \delta(\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0}) - \mathbf{c}) \mathcal{D}\mathbf{u}},$$

where $\mathcal{F}[\mathbf{u}]$ is the functional probability distribution of the velocity field, and the integrations are meant to be functional integrations. In the Appendix the asymptotic form of (13) valid for large c is obtained to be

$$(14) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) \rightarrow \Delta_{lp}^{-1} \Delta_{mq}^{-1} c_p c_q \int \Sigma_{il}(\mathbf{R}, \mathbf{x}) \Sigma_{jm}(\mathbf{R}, \mathbf{x}') \varphi(\mathbf{x}) \varphi(\mathbf{x}') d^3 x d^3 x',$$

where

$$(15) \quad \Delta_{ij} = \int \Sigma_{ij}(\mathbf{x}, \mathbf{x}') \varphi(\mathbf{x}) \varphi(\mathbf{x}') d^3 x d^3 x',$$

$$(16) \quad \Sigma_{ij}(\mathbf{x}, \mathbf{x}') = \langle u_i(\mathbf{x}) u_j(\mathbf{x}') \rangle = \langle u_i(\mathbf{x}) u_j(\mathbf{x}) \rangle - \frac{1}{2} \int c_i c_j \mathcal{P}(\mathbf{c}, \mathbf{x} - \mathbf{x}') d^3 c,$$

and $\varphi(\mathbf{x}) \equiv \delta(\mathbf{x} - \mathbf{r}) - \delta(\mathbf{x})$.

The asymptotic form of \mathbf{p} , obtained from (10) using (14), turns out to be a rather complicated functional of $\mathcal{P}(\mathbf{c}, \mathbf{r})$. In order to simplify the resulting expression we will use the inertial range form (Landau and Lifshitz, 1959)

$$(17) \quad \int c_i c_j \mathcal{P}(\mathbf{c}, \mathbf{r}) d^3 c = K r^\zeta \left(\delta_{ij} - \frac{\zeta}{\zeta + 2} \frac{r_i r_j}{r^2} \right),$$

where K and ζ are constant, with $\zeta = 2/3$ for Kolmogorov 1941 scaling (K41) (Kolmogorov, 1941). When (17) is used in (16), (10) can be explicitly evaluated. After a very long calculation one obtains

$$(18) \quad p_k = \frac{c_i c_j}{r} [A h_i h_j h_k + B h_k \delta_{ij} + C (h_i \delta_{jk} + h_j \delta_{ik})],$$

with $h_i = r_i/r$, and where A, B and C are dimensionless constants obtained numerically which depend only on ζ . For K41 scaling A, B and C are

$$(19) \quad A = 0.0156, \quad B = -0.2510, \quad \text{and} \quad C = 0.0740,$$

where the errors are of one unit in the last figure.

Having obtained the asymptotic form of \mathbf{p} (18) one can now propose a quite general expression valid for all c

$$(20) \quad p_k = a_{ijk} c_i c_j + b_{ik} c_i + d_k,$$

that has the correct asymptotic form, and where b_{ik} and d_k depend only on \mathbf{r} . To determine these coefficients we will use the following identities satisfied by \mathbf{p} (Kármán and Howarth, 1938)

$$(21) \quad \left\langle \frac{\partial p}{\partial x_k} \right\rangle = \rho \int p_k \mathcal{P}(\mathbf{c}, \mathbf{r}) d^3 c = 0,$$

and

$$(22) \quad \left\langle \frac{\partial p}{\partial x_k} (u_l(\mathbf{x} + \mathbf{r}) - u_l(\mathbf{x})) \right\rangle = \rho \int p_k c_l \mathcal{P}(\mathbf{c}, \mathbf{r}) d^3 c = 0.$$

The coefficients in (20) are easily obtained to be

$$(23) \quad b_{ik} = -a_{lmk} \langle \Delta u_l \Delta u_m \Delta u_p \rangle \langle \Delta u_p \Delta u_i \rangle^{-1},$$

$$(24) \quad d_k = -a_{lmk} \langle \Delta u_l \Delta u_m \rangle.$$

Again, a simpler version can be obtained using the inertial range expressions (Landau and Lifshitz, 1959)

$$(25) \quad \langle \Delta u_l \Delta u_m \rangle = \frac{4}{3} B_{rr} \left(\delta_{lm} - \frac{1}{4} h_l h_m \right),$$

$$(26) \quad \langle \Delta u_l \Delta u_m \Delta u_p \rangle = \frac{1}{3} B_{rrr} (h_l \delta_{mp} + h_m \delta_{lp} + h_p \delta_{lm}),$$

where $B_{rr} = \langle (\Delta u_r)^2 \rangle$ and $B_{rrr} = \langle (\Delta u_r)^3 \rangle$, although we will not write them explicitly at this point.

4. Dissipation term

We now proceed to model the conditional dissipation. For this we consider that essentially all dissipation takes place in structures of Lundgren's type (Lundgren, 1982; Lundgren, 1993) characterized by a spiral flow around an axis taken as z , with a stretching flow of cylindrical components

$$(27) \quad u_z = az,$$

$$(28) \quad u_r = -\frac{1}{2} ar,$$

and an induced azimuthal flow $u_\varphi(r, \varphi)$. In practice there is an induced radial flow of smaller magnitude than that in (28) (Pullin and Saffman, 1993), which we neglect in the estimations to follow. We consider now that the points $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_2 = \mathbf{x} + \mathbf{r}$ are inside one such vortex and denote its position and orientation by giving the distance to the vortex axis d_0 and height z_0 of the midpoint $\mathbf{x} + \mathbf{r}/2$, the angle θ_0 between \mathbf{r} and the vortex axis, and the angle φ_0 between the component of \mathbf{r} in the plane perpendicular to the axis and the segment determined by the midpoint and the point z_0 of the axis. With these conventions the components parallel and perpendicular to \mathbf{r} of the difference of velocities can be written as

$$(29) \quad \Delta u_r = (u_\varphi(d_1, \varphi_1) \sin \gamma_1 - u_\varphi(d_2, \varphi_2) \sin \gamma_2) \sin \theta_0 + ar (\cos^2 \theta_0 - 1/2 \sin^2 \theta_0),$$

$$(30) \quad (\Delta u_\perp)^2 = (u_\varphi(d_1, \varphi_1) \cos \gamma_1 - u_\varphi(d_2, \varphi_2) \cos \gamma_2)^2 + \sin^2 \theta_0 (3/2 ar \cos \theta_0 - u_\varphi(d_1, \varphi_1) \sin \gamma_1 + u_\varphi(d_2, \varphi_2) \sin \gamma_2)^2$$

where d_1 and d_2 are the distances of points \mathbf{x}_1 and \mathbf{x}_2 to the axis, given by

$$d_1^2 = d_0^2 + (r/2 \sin \theta_0)^2 - d_0 r \sin \theta_0 \cos \varphi_0,$$

$$d_2^2 = d_0^2 + (r/2 \sin \theta_0)^2 + d_0 r \sin \theta_0 \cos \varphi_0,$$

φ_1 and φ_2 are the azimuthal coordinate of those points (the explicit expression will not be given), and γ_1 and γ_2 are defined by

$$\cos \gamma_1 = (d_0 \cos \varphi_0 - r/2 \sin \theta_0)/d_1,$$

$$\cos \gamma_2 = (d_0 \cos \varphi_0 + r/2 \sin \theta_0)/d_2.$$

The idea is that, given the values of Δu_r and Δu_\perp , we should identify all structures that could be responsible for these values, determine the dissipation in each of them and average properly. This is certainly a formidable task and many approximations are required. To begin with, we will assume that the azimuthal flow is mainly of the form $u_\varphi(r) = \Gamma/(2\pi r)$, which reduces expressions (29) and (30) to

$$(31) \quad \Delta u_r = \frac{\Gamma}{2\pi d_1^2 d_2^2} d_0^2 r \sin^2 \theta_0 \sin 2\varphi_0 + ar (\cos^2 \theta_0 - 1/2 \sin^2 \theta_0),$$

$$(32) \quad (\Delta u_\perp)^2 = \left(\frac{\Gamma r \sin \theta_0}{2\pi d_2^2 d_1^2} \right)^2 [d_0^2 (\sin^2 \varphi_0 - \cos^2 \varphi_0) + r^2/4 \sin^2 \theta_0]^2 + \cos^2 \theta_0 \sin^2 \theta_0 \left(3/2 ar - \frac{\Gamma d_0^2 r \sin 2\varphi_0}{2\pi d_1^2 d_2^2} \right)^2$$

In order to simplify the derivations and at the same time be certain that both \mathbf{x}_1 and \mathbf{x}_2 are well inside the vortex structure, only those cases with $d_0 \ll r$ will be considered. This leads to

$$(33) \quad a = \frac{\Delta u_r}{r (\cos^2 \theta_0 - 1/2 \sin^2 \theta_0)},$$

$$(34) \quad \Gamma = \pm \pi |r/2 \sin \theta_0| \left[(\Delta u_{\perp})^2 - (\Delta u_r)^2 \frac{3/2 \cos^2 \theta_0 \sin^2 \theta_0}{(\cos^2 \theta_0 - 1/2 \sin^2 \theta_0)^2} \right]^{1/2}.$$

In order to average over all possible structures we should know the distribution of stretching rates a , this is certainly very difficult and to circumvent this point we will consider only the asymptotic form of the conditional dissipation for very large values of Δu_r . From (33) this involves structures with large values of a , which we assume to decrease in number sufficiently rapidly as a increases, so that the main contribution to the conditioned dissipation comes from those structures with the minimum value of a compatible with (33). This means taking $\theta_0 \simeq \pi/2$ when $\Delta u_r < 0$, and $\theta_0 \simeq 0$ when $\Delta u_r > 0$ (in fact, θ_0 cannot be very close to zero because in this case \mathbf{x}_1 and \mathbf{x}_2 would lie in the core of the vortex where the assumed form of the azimuthal velocity does not hold). That is

$$(35) \quad a = \frac{\Delta u_r}{r} \quad \text{if } \Delta u_r > 0, \quad \text{and} \quad a = -\frac{2\Delta u_r}{r} \quad \text{if } \Delta u_r < 0.$$

To estimate the dissipation in one such structure we now follow Gilbert's ideas (Gilbert, 1993) and consider the vorticity at \mathbf{x}_1 in scales of order r , $\omega_{(r)}$. This is the large scale vorticity, stretching increases this value at decreasing scales as $\omega \sim \omega_{(r)} \exp(at)$, this proceeds until the dissipation scale η is reached. Since a structure of initial scale r develops small scales (by stretching and axisymmetric windup) as $l \sim ra/\sigma \exp(-3at/2)$, the increase in vorticity proceeds up to a time given by

$$(36) \quad t_M \sim \frac{2}{3a} \ln \left(\frac{ar}{\sigma\eta} \right).$$

Here σ is a measure of the differential rotation of the flow at \mathbf{x}_1 which can be estimated as $\omega_{(r)}$ itself. The instantaneous viscous dissipation at \mathbf{x}_1 can then be approximated by

$$(37) \quad \varepsilon \simeq \nu \omega^2 \simeq \nu \omega_{(r)}^2 \exp(2at).$$

The average dissipation is now obtained by averaging over all possible values of ε in the different vortices, which can be translated in the time average over one such vortex

$$(38) \quad \langle \varepsilon | r, \Delta u_r, \Delta u_{\perp} \rangle \simeq \frac{1}{t_M} \int_0^{t_M} \varepsilon dt \simeq \frac{3\nu\omega_{(r)}^2}{4 \ln(ar/\sigma\eta)} \left(\frac{ar}{\sigma\eta} \right)^{4/3}.$$

The estimation of $\omega_{(r)}$ in the asymptotic limit considered can be made as $\omega_{(r)} \sim \Delta u_r/r$ (if instead $\omega_{(r)} \sim \Delta u_{\perp}/r$ were used, after proper averaging over Δu_{\perp} the final result would be the same). With these considerations

$$(39) \quad \langle \varepsilon | r, \Delta u_r, \Delta u_{\perp} \rangle \simeq \frac{3C [\text{sign}(\Delta u_r)] \langle \varepsilon \rangle (\Delta u_r)^2}{4 (\langle \varepsilon \rangle \eta)^{2/3} (r/\eta)^{2/3} \ln(r/\eta)},$$

where use of the relation $\eta = (\nu^3/\langle \varepsilon \rangle)^{1/4}$ was made, and where

$$C(1) \simeq 1 \quad \text{and} \quad C(-1) \simeq 2^{4/3}.$$

5. Reduced pdf

The equation for the pdf of Δu_r , is finally obtained by integrating out the perpendicular velocity difference, $c_\perp = \sqrt{c^2 - c_\parallel^2}$, in (5), with $c_\parallel = \mathbf{c} \cdot \mathbf{r}/r$. The result is

$$(40) \quad \begin{aligned} \frac{\partial P}{\partial t} = & -\xi \frac{\partial P}{\partial r} - 2r^{-1} \xi P - r^{-1} \frac{\partial \Phi}{\partial \xi} \\ & - \frac{\partial}{\partial \xi} (p_\parallel P) - \frac{\partial^2}{\partial \xi^2} (DP) \\ & + 2\nu r^{-2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial P}{\partial r} \right) + 2 \frac{\partial}{\partial \xi} (\xi P) + \frac{\partial^2 \Phi}{\partial \xi^2} \right], \end{aligned}$$

where for easy of notation the definition $\xi \equiv c_\parallel$ was used, $P = \int \mathcal{P} d^2 c_\perp$, and $\Phi = \int c_\perp^2 \mathcal{P} d^2 c_\perp$. Moreover,

$$(41) \quad p_\parallel = \frac{1}{r} \left[A' \xi^2 + B \frac{\Phi}{P} - (A' + 2B/3) \frac{B_{rrr}}{B_{rr}} \xi - (A' + 8B/3) B_{rr} \right],$$

where $A' = A + B + 2C$. In (41) expression (20) was used, together with (23), (24), (25), and (26).

The dissipation term is

$$(42) \quad D = 2 \langle \varepsilon_\parallel | \xi, r \rangle,$$

where $\varepsilon_\parallel = \nu \nabla u_\parallel \cdot \nabla u_\parallel$, with $u_\parallel = \mathbf{u} \cdot \mathbf{r}/r$. Note that $\int \langle \varepsilon_\parallel | \xi, r \rangle P d\xi = \langle \varepsilon \rangle / 3$, so that one can write

$$(43) \quad \int DP d\xi = \frac{2}{3} \langle \varepsilon \rangle.$$

From the asymptotic expression (39) D could be written as (Kraichnan, 1994)

$$D = a_0(r) + a_1(r) \xi + a_2(r) \xi^2,$$

where the a 's have in general different expressions according to whether ξ is positive or negative. Positiveness of D further requires that

$$(44) \quad a_0(r) > 0, \quad a_2(r) > 0, \quad \text{and} \quad a_1^2(r) < 4 a_0(r) a_2(r).$$

By the normalization condition (43) D can then be conveniently written as

$$(45) \quad D = \frac{2}{3} \langle \varepsilon \rangle [b_0(r) + b_1(r) R_1^{-1} \xi + b_2(r) B_{rr}^{-1} \xi^2],$$

where $R_1(r) \equiv \langle |\Delta u_\parallel| \rangle = \int |\xi| P d\xi$, and the $b(r)$'s are dimensionless functions of r , with $b_0 + b_2 = 1$. To further justify an expression like (45) consider that for $r < \eta$ the velocity field is a smooth function of \mathbf{x} and so, by expansion of u_\parallel (Stolovitzky and Sreenivasan, 1993), it is readily seen that $D \approx \langle \varepsilon \rangle S_2^{-1}(r) \xi^2$. On the other hand, for r larger than the outer scale L , which determines the largest correlation length in the flow, D should be independent of ξ . A linear form for the intermediate region results naturally if one looks for analytic expressions at $\xi = 0$. The $b(r)$'s mostly determine the transition between the different regimes. Models like

(45), with simple algebraic dependence on ξ , are well satisfied in the case of a passive scalar (Sinai and Yakhot, 1989; Vaienti *et al.*, 1994; Kraichnan *et al.*, 1995; Ching, 1996).

Using $\langle \varepsilon_{||} | \xi, r \rangle \simeq \langle \varepsilon | \xi, r \rangle / 3$, expression (39) indicates

$$(46) \quad b_2(r) = \frac{3 C_2 C [\text{sign}(\xi)]}{4 \ln(r/\eta)},$$

where the K41 expression $B_{rr} = C_2 (\langle \varepsilon \rangle r)^{2/3}$ was used.

6. Asymptotic solutions

To study the solutions of (40) one needs a relationship between Φ and P , and the explicit form of $b_1(r)$. We now show that once Φ is determined, it is possible to study quite generally various asymptotic solutions of (40) without the explicit expression of $b_1(r)$. To begin with, consider that, by its definition, Φ satisfies

$$(47) \quad \int_{-\infty}^{+\infty} \Phi d\xi = \langle (\Delta u_{\perp})^2 \rangle = \frac{8}{3} B_{rr},$$

$$(48) \quad \int_{-\infty}^{+\infty} \xi \Phi d\xi = \langle \Delta u_r (\Delta u_{\perp})^2 \rangle = \frac{2}{3} B_{rrr},$$

where the last equality in each expression holds in the inertial range for K41 scaling (Landau and Lifshitz, 1959). It is reasonable to expect that Φ is not of higher order in ξ than $\xi^2 P$. With this assumption, the expression that preserves positiveness, and satisfies (47) and (48) is

$$(49) \quad \Phi = \left(2 B_{rr} + \frac{2}{3} \xi^2 \right) P.$$

With expression (49), (41) reduces to

$$(50) \quad p_{||} = \frac{\alpha}{r} \left[\xi^2 - \frac{B_{rrr}}{B_{rr}} \xi - B_{rr} \right],$$

with $\alpha = A' + 2B/3$, for K41 scaling (19) give $\alpha = -0.255$.

When (49) and (50) are used in (40) the resulting equation, although linear, is rather complex, and to start simplifying it we consider only stationary solutions. For stationary solutions to exist, one must assume an appropriate forcing. As is usual, we will assume a random force with Gaussian distribution, delta-correlated in time, and with spatial correlation length L , the outer scale. It can be seen in this case, using the formalism developed by Novikov (Novikov, 1964), that this leads to a term of the form $F(r/L) \partial^2 P / \partial \xi^2$ in the right-hand side of (40), where F does not depend on ξ . The effect of the forcing amounts then to a redefinition of $b_0(r)$ in (45), which will be shown not to contribute to the asymptotic solutions to be considered. Two interesting limits can now be analyzed relatively simply. The first one corresponds to $|\xi| \rightarrow \infty$ with r large but finite, the latter assumption allows to neglect the viscous term in (40) which can be justified *a posteriori*. The asymptotic equation for P then reads

$$(51) \quad \xi \frac{\partial P}{\partial r} + \frac{2\xi}{r} P = -\frac{2/3 + \alpha}{r} \frac{\partial}{\partial \xi} (\xi^2 P) - \frac{2}{3} \langle \varepsilon \rangle \frac{b_2(r)}{B_{rr}(r)} \frac{\partial^2}{\partial \xi^2} (\xi^2 P).$$

If one seeks asymptotic solutions of the stretched exponential type (Tabeling *et al.*, 1996)

$$(52) \quad P \sim \exp(-f(r) |\xi|^\lambda),$$

it is readily seen that $\lambda = 1$, and that $f(r)$ satisfies

$$(53) \quad \frac{df}{dr} = -(2/3 + \alpha) \frac{f}{r} + \text{sign}(\xi) \frac{2}{3} \langle \varepsilon \rangle \frac{b_2(r)}{B_{rr}(r)} f^2.$$

If $r \rightarrow \infty$, the last term in the right hand side of (53) can be neglected and exponential tails exist for both signs of ξ , with

$$(54) \quad f(r) \sim r^{-(2/3+\alpha)}.$$

If r is not large, no term can be neglected in (53) and the exponential solution can exist only for $\xi > 0$. Praskovsky and Oncley (Praskovsky and Oncley, 1994) found experimentally that the asymptotic pdf at Reynolds numbers can be well approximated by

$$(55) \quad P \rightarrow \exp(-K [\text{sign}(\xi)] (r/\eta)^\beta |\xi|/\sigma_{\Delta u}),$$

where $\sigma_{\Delta u} \equiv \sqrt{B_{rr}(r)}$, $\beta \simeq 0.17$, and $K(-1) \simeq 0.51$, $K(1) \simeq 0.58$. The range of r tested was moderately large: $30 < r/\eta < 3000$. In this range the logarithm appearing in (46) can be very well approximated by $\ln(x) \simeq 2.1 x^{0.17}$, with an error below 10% for $30 < x < 60$, and below 3% for $60 < x < 3000$. This allows to easily solve (53) for $\xi > 0$ to obtain

$$(56) \quad f(r) = 4.2 \frac{\alpha + 0.5}{C(1)} C_2^{1/2} (r/\eta)^{0.17} / \sigma_{\Delta u}.$$

With $C_2 \simeq 1.8$, $\alpha = -0.255$, and $C(1) = 1$, expression (55) results for $\xi > 0$ with $K(1) \simeq 1.4$, in reasonable agreement with the experimental value considering that $C(1) = 1$ is only a lowest bound estimation.

For $\xi < 0$ there is not exponentially decaying solution but rather a power-law decaying solution which is easily obtained to be

$$(57) \quad P \rightarrow ((r/\eta)^{\zeta_2/2-\mu} |\xi|/\sigma_{\Delta u})^{-\lambda},$$

with $\lambda = 2(\alpha + 5/3)/(2/3 + \alpha - \mu)$. The value of μ is not determined but just restricted to be less than $2/3 + \alpha$. If one uses the experimental information that P can be well represented as a function of the variable in (55), it results that $\mu = \zeta_2/2 - 0.17 \simeq 0.163$, and $\lambda \simeq 11$ if $\alpha = -0.255$. This large value of λ does not allow to easily discriminate between a power-law and an exponential decay, in fact, (57) with the discussed values for the parameters fits very well the observed data. Solution (57) is actually valid for both signs of ξ , and is the leading solution at large values of $|\xi|$, when r is not too large. The transition from the exponential to the algebraic behaviour is expected to occur for r/η less than roughly $5 \cdot 10^3$, for in this range it can be estimated, using (54), that the last term in the right-hand side of (53) is only marginally smaller than the first one. Theoretically, pdf's with algebraic tails have been obtained for negative velocity differences in the case of Burgers turbulence, see (Bouchaud and Mézard, 1996; Weinan *et al.*, 1997) and references therein, for the velocity gradient in two-dimensional turbulence (Jiménez, 1996), and for the passive scalar (Sinai and Yakhot, 1989; Valiño *et al.*, 1994).

The other limit is $r \rightarrow \infty$, with $|\xi|$ large but finite. In this case the equation cannot be simplified *a priori*, and the whole expression must be considered, so we will not write it down but just quote the results. Tabeling and coworkers (Tabeling *et al.*, 1996) found experimentally that the asymptotic pdf at high Reynolds numbers could be well approximated by

$$(58) \quad P \rightarrow \exp[-K(|\xi|/\sigma_{\Delta u})^\gamma],$$

where K and γ both depend on r . The exponent γ increases from about 0.6 at the lower limit of the inertial range, until it saturates at the value of 2, the Gaussian limit, for large separations. On the other hand, K decreases with increasing r . With this behaviour in mind, we first explore the possibility of Gaussian solutions of (40), neglecting again the viscous term in the limit considered, but retaining the other terms. By direct substitution it can be easily shown that a Gaussian pdf of the form

$$(59) \quad P \rightarrow \exp[-K(r)(\xi/\sigma_{\Delta u})^2]$$

exists only for $|\xi|/\sigma_{\Delta u} \ll O(1) \ln(r/\eta)$, where $O(1)$ refers to the order in r/η . In this case, $K(r)$ is of the form $K(r) \sim r^{-0.157}$ in good agreement with the experimentally determined behaviour shown in this region. For larger values of $|\xi|$, always in the limit $r \rightarrow \infty$, the solution cannot be Gaussian but rather of the exponential form, as found above (54),

$$(60) \quad P \rightarrow \exp[-K(r)|\xi|/\sigma_{\Delta u}],$$

with $K(r) \sim r^{-0.078}$, in agreement with the behaviour found experimentally. Interestingly enough, these asymptotic solutions can be obtained without reference to the explicit form of $b_1(r)$, only considering that it is a decreasing function of r as it must be by conditions (44). This range, $r \rightarrow \infty$, and $|\xi|/\sigma_{\Delta u} > \ln(r/\eta)$, seems to be that studied with alternative techniques which lead to exponential in $|\xi|$ asymptotic decay (Kraichnan, 1990; Giles, 1995).

The behaviour represented by (55) seems at first glance contradictory with that of (58). However, in (Praskovsky and Oncley, 1994) the range of r tested was small, compared to that in (Tabeling *et al.*, 1996), but the velocity difference was somewhat larger. In this sense, we argue that the limit $|\xi| \rightarrow \infty$, with r large is more pertinent to (Praskovsky and Oncley, 1994), and the other limit studied is more consistent with (Tabeling *et al.*, 1996). In fact, as r decreases one moves from one limit to the other, and the observed decrease of the exponent γ in (58) can be interpreted as the approach to the power-law behaviour.

7. Conclusions

In conclusion, an equation for the pdf of longitudinal velocity difference is presented which is shown to have asymptotic solutions consistent with experiments. The equation relies on the modeling of conditioned averages of pressure gradient and viscous dissipation. The model for the conditioned average of pressure gradient was derived using controlled expansion techniques and tensor identities which makes it very reliable. An asymptotic model for the conditioned average of viscous dissipation, on the other hand, was constructed using a more phenomenological approach, which is based on the properties of dissipative structures with well established pertinence to real turbulence (Pullin and Saffman, 1993; Segel, 1995). We think it to be a good first approximation, considering the good quantitative agreement with experiment, equation (56), but with much work still required. In this respect, experimental and numerical evaluation of expressions like (45) for the

conditional dissipation is certainly difficult but worth doing. In particular, it is possible that expression (45) is not correct for very large values of ξ , because it would require arbitrarily large values of the stretching rate a in (35), which might be impossible in a given flow. In this case, the dissipation associated to very large ξ is probably related to complex interactions of different structures, which can then provide that value of ξ , with a ξ dependence different from that in (45). For instance, it can be seen that if the conditional dissipation is independent of ξ at very large ξ , the far asymptotic tails are either Gaussian or decay faster than Gaussian, as indicated by general theoretical arguments and experimental evidence (Noullez *et al.*, 1997). Moreover, in this case the external forcing considered above cannot be neglected and these tails are then non-universal. The asymptotic pdf's obtained in this work should then be considered as intermediate asymptotic solutions.

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APPENDIX

In order to evaluate the integrals in (13) in the $c \rightarrow \infty$ limit, we will develop them in an infinite series, each term of which will be shown to be decomposable into three sub-terms of distinct order in c . Finally, by retaining the highest order sub-term, the series will be summed up. To begin with, $\mathcal{F}[\mathbf{u}]$ will be written as

$$(A.1) \quad \mathcal{F}[\mathbf{u}] = \frac{1}{Z} \mathcal{F}_0[\mathbf{u}] \mathcal{G}[\mathbf{u}],$$

where $\mathcal{F}_0[\mathbf{u}] = Z_0^{-1} \exp\left(-\frac{1}{2} \int \sigma_{lm}(\mathbf{x}, \mathbf{x}') u_l(\mathbf{x}) u_m(\mathbf{x}') d^3x d^3x'\right)$, Z_0 is a normalization constant, and $\sigma_{lm}(\mathbf{x}, \mathbf{x}')$ is to be determined later. $\mathcal{G}[\mathbf{u}]$ is the functional defined by the very expression (A.1) and the normalization constant is given by

$$(A.2) \quad Z = \int \mathcal{F}_0[\mathbf{u}] \mathcal{G}[\mathbf{u}] \mathcal{D}\mathbf{u}.$$

Since, by definition, $\mathcal{G}[\mathbf{u}]$ must be regular at $\mathbf{u} = \mathbf{0}$, we Taylor expand it as

$$(A.3) \quad \mathcal{G}[\mathbf{u}] = \sum_{n=0}^{\infty} \frac{1}{n!} K(1, \dots, n) u(1) \dots u(n).$$

Here, for easy of notation, tensor indices and spatial variables are represented by a single numerical variable, whenever possible, such as, for instance $u(1) = u_{i_1}(\mathbf{x}_1)$, $\sigma(1, 2) = \sigma_{i_1 i_2}(\mathbf{x}_1, \mathbf{x}_2)$, etc. Moreover, contraction of repeated tensor indices and simultaneous integration over corresponding spatial variables is implicit:

$$(A.4) \quad K(1, \dots, n) u(1) \dots u(n) = \int K_{i_1 \dots i_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) u_{i_1}(\mathbf{x}_1) \dots u_{i_n}(\mathbf{x}_n) d^3x_1 \dots d^3x_n.$$

We turn now to the evaluation of (13). To do this we use the Fourier expansion of Dirac's delta function

$$(A.5) \quad \delta(\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0}) - \mathbf{c}) = \frac{1}{(2\pi)^3} \int \exp[i\mathbf{k} \cdot (\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0}) - \mathbf{c})] d^3k,$$

to write:

$$(A.6) \quad \mathcal{P}(\mathbf{c}, \mathbf{r}) = \int \mathcal{F}[\mathbf{u}] \delta(\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{0}) - \mathbf{c}) \mathcal{D}\mathbf{u} = \frac{1}{(2\pi)^3} \int d^3k \exp(-i\mathbf{k} \cdot \mathbf{c}) \\ \frac{1}{Z} \int \mathcal{F}_0[\mathbf{u}] \mathcal{G}[\mathbf{u}] \exp \left[i \int k_l \varphi(\mathbf{x}) u_l(\mathbf{x}) d^3x \right] \mathcal{D}\mathbf{u},$$

where $\varphi(\mathbf{x}) \equiv \delta(\mathbf{x} - \mathbf{r}) - \delta(\mathbf{x})$. It is now useful to introduce an auxiliary functional given by

$$(A.7) \quad B[\mathbf{J}] = \frac{1}{(2\pi)^3} \int d^3k \exp(-i\mathbf{k} \cdot \mathbf{c}) \\ \frac{1}{Z} \int \mathcal{F}_0[\mathbf{u}] \exp \left[J(1) u(1) + i \int k_l \varphi(\mathbf{x}) u_l(\mathbf{x}) d^3x \right] \mathcal{D}\mathbf{u},$$

which allows to write (A.6) as

$$(A.8) \quad \mathcal{P}[\mathbf{c}, \mathbf{r}] = \sum_n \frac{1}{n!} K(1, \dots, n) \left\{ \frac{\delta^n B}{\delta J(1) \dots \delta J(n)} \right\}_{\mathbf{J}=0},$$

and

$$(A.9) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) \mathcal{P}(\mathbf{c}, \mathbf{r}) = \left\{ \frac{\delta^2}{\delta J_i(\mathbf{R}) \delta J_j(\mathbf{R})} \sum_n \frac{1}{n!} K(1, \dots, n) \frac{\delta^n B}{\delta J(1) \dots \delta J(n)} \right\}_{\mathbf{J}=0}.$$

The functional integral in (A.7) is of the Gaussian type and can then be performed (Popov, 1987), leading to a Gaussian type of integral in \mathbf{k} which can then also be calculated to give

$$(A.10) \quad B[\mathbf{J}] = \frac{Z_0}{Z} (\det \Delta)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \Delta_{ij}^{-1} c_i c_j \right) \exp W[\mathbf{J}],$$

where

$$(A.11) \quad \Delta_{ij} = \int \Sigma_{ij}(\mathbf{x}, \mathbf{x}') \varphi(\mathbf{x}) \varphi(\mathbf{x}') d^3x d^3x',$$

Σ is the inverse of σ in the sense that

$$(A.12) \quad \Sigma(1, 3) \sigma(3, 2) = \sigma(1, 3) \Sigma(3, 2) = \delta_{i_1 i_2} \delta(\mathbf{x}_1 - \mathbf{x}_2),$$

and

$$(A.13) \quad W[\mathbf{J}] = \frac{1}{2} [\Sigma(1, 2) J(1) J(2) - \Delta_{ij}^{-1} b_i b_j + \Delta_{ij}^{-1} (b_i c_j + b_j c_i)],$$

with

$$(A.14) \quad b_i[\mathbf{J}] = \int \Sigma_{il}(\mathbf{x}, \mathbf{x}') \varphi(\mathbf{x}) J_l(\mathbf{x}') d^3x d^3x'$$

a linear functional of \mathbf{J} . By differentiating (A.10) one can write

$$(A.15) \quad \frac{\delta^2 B}{\delta J_i(\mathbf{R}) \delta J_j(\mathbf{R})} = \Theta_{ij}[\mathbf{J}] B,$$

where

$$(A.16) \quad \Theta_{ij}[\mathbf{J}] = \frac{\delta^2 W}{\delta J_i(\mathbf{R}) \delta J_j(\mathbf{R})} + \frac{\delta W}{\delta J_i(\mathbf{R})} \frac{\delta W}{\delta J_j(\mathbf{R})}.$$

We will not write the explicit expression of $\Theta_{ij}[\mathbf{J}]$ (easily obtained from (A.13) but quite cumbersome) but just point out its pertinent properties:

i) It is a quadratic functional of \mathbf{J} and a quadratic function of \mathbf{c} .

ii) Every derivative of $\Theta_{ij}[\mathbf{J}]$ with respect to \mathbf{J} decreases one order in \mathbf{c} , in contrast, every derivative of B increases one order in \mathbf{c} .

With these considerations we now examine a generic derivative in the sum over n in (A.9) and write in a rather symbolic notation

$$(A.17) \quad \frac{\delta^2}{\delta J_i(\mathbf{R}) \delta J_j(\mathbf{R})} \left[\frac{\delta^n B}{\delta J(1) \dots \delta J(n)} \right] = \sum_{m=0}^n \binom{n}{m} \frac{\delta^{n-m} B}{\delta J^{n-m}} \frac{\delta^m \Theta_{ij}}{\delta J^m}.$$

By property i) the sum over m runs only from 0 to 2, and by property ii) the $m = 0$ term is two orders in \mathbf{c} higher than the $m = 1$ term and four orders higher than the $m = 2$ term. This means that the asymptotic expression of $\Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r})$ for $c \rightarrow \infty$ can be determined by retaining the most divergent, $m = 0$ term for every n in (A.9) to write

$$(A.18) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) \mathcal{P}(\mathbf{c}, \mathbf{r}) \rightarrow \Theta_{ij}[\mathbf{J} = 0] \sum_n \frac{1}{n!} K(1, \dots, n) \left\{ \frac{\delta^n B}{\delta J(1) \dots \delta J(n)} \right\}_{\mathbf{J}=0}.$$

By comparing (A.18) with (A.8) we see immediately that

$$(A.19) \quad \Gamma_{ij}(\mathbf{R}, \mathbf{c}, \mathbf{r}) \rightarrow \Theta_{ij}[\mathbf{J} = 0].$$

The remarkable point about (A.19) is that the large c asymptotic of Γ is the same that would result if the functional $\mathcal{F}[\mathbf{u}]$ were Gaussian. Of course, if $\mathcal{F}[\mathbf{u}]$ were effectively Gaussian, (A.19) would hold for all c and not only asymptotically. Note however that the full non-Gaussian factor $\mathcal{G}[\mathbf{u}]$ was retained in the derivation of (A.19). These considerations allow to determine the unknown function σ (and consequently Σ by (A.12)) by simply noting that if $\mathcal{F}[\mathbf{u}]$ were Gaussian then it should coincide with $\mathcal{F}_0[\mathbf{u}]$, for which it is easy to show (Popov, 1987) that

$$(A.20) \quad \langle u(1) u(2) \rangle = Z_0^{-1} \int \exp \left[-\frac{1}{2} \sigma(3, 4) u(3) u(4) \right] u(1) u(2) \mathcal{D} \mathbf{u} = \Sigma(1, 2),$$

where $\langle u(1) u(2) \rangle$ is the second order velocity correlation. With this, and retaining the highest \mathbf{c} contributions in $\Theta_{ij}[\mathbf{J} = 0]$ we finally obtain (14) in the main text.

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